

On the homology of the spaces of long knots

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Abstract

Keywords: discriminant of the space of knots, bialgebra of chord diagrams, Hochschild complex, operads of Poisson – Gerstenhaber – Batalin-Vilkovisky algebras.

This paper is a more detailed version of [T1], where the first term of the Vassiliev spectral sequence (computing the homology of the space of long knots in \mathbb{R}^d , $d \geq 3$) was described in terms of the Hochschild homology of the Poisson algebras operad for d odd, and of the Gerstenhaber algebras operad for d even. In particular, the bialgebra of chord diagrams arises as some subspace of this homology. The homology in question is the space of characteristic classes for Hochschild cohomology of Poisson (resp. Gerstenhaber) algebras considered as associative algebras. The paper begins with necessary preliminaries on operads.

Also we give a simplification of the computations of the first term of the Vassiliev spectral sequence.

We do not give proofs of the results.

0 Introduction

First we recall some known facts on the Vassiliev spectral sequence and then proceed to explaining of the main idea of the work.

0.1

Let us fix a non-trivial linear map $l : \mathbb{R}^1 \hookrightarrow \mathbb{R}^d$. We will consider the *space of long knots*, *i. e.*, of injective smooth non-singular maps $\mathbb{R}^1 \hookrightarrow \mathbb{R}^d$, that coincide with the map l outside some compact set (this set is not fixed). The long knots form an open everywhere dense subset in the affine space \mathcal{K} of all smooth maps $\mathbb{R}^1 \rightarrow \mathbb{R}^d$ with the same behavior at infinity. The complement $\Sigma \subset \mathcal{K}$ of this dense subset is called the *discriminant space*. It consists of the maps having self-intersections or singularities. Any cohomology class $\gamma \in H^i(\mathcal{K} \setminus \Sigma)$ of the knot space can be realized as the linking coefficient with an appropriate chain in Σ of codimension $i + 1$ in \mathcal{K} .

Following [V5] we will assume that the space \mathcal{K} has a very large but finite dimension ω . A partial justification of this assumption uses finite dimensional approximations of \mathcal{K} , see [V1]. Below we indicate by quotes non-rigorous assertions using this assumption and needing a reference to [V2] for such a justification.

The main tool of Vassiliev's approach to computation of the (co)homology of the knot space is the *simplicial resolution* σ (constructed in [V1]) of the discriminant Σ . This resolution is also called the *resolved discriminant*. The natural projection $\Pi : \bar{\sigma} \rightarrow \bar{\Sigma}$ is a “homotopy equivalence” between the “one-point compactifications” of the spaces σ and Σ . By the “Alexander duality”, the reduced homology groups $\tilde{H}_*(\bar{\sigma}, \mathbb{k}) \equiv \tilde{H}_*(\bar{\Sigma}, \mathbb{k})$ of these compactifications “coincide” (up to a change of dimension) with the cohomology groups of the space of knots:

$$\tilde{H}^i(\mathcal{K} \setminus \Sigma, \mathbb{k}) \simeq \tilde{H}_{\omega-i-1}(\bar{\Sigma}, \mathbb{k}) \equiv \tilde{H}_{\omega-i-1}(\bar{\sigma}, \mathbb{k}), \quad (0.1)$$

where \mathbb{k} is a commutative ring of coefficients.

In the space σ there is a natural filtration

$$\emptyset = \sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \dots \quad (0.2)$$

Conjecture 0.3 *The spectral sequence (called Vassiliev's main spectral sequence) associated with the filtration (0.2) and computing the “Borel-Moore homology groups of the resolution σ ” stabilizes over \mathbb{Q} in the first term. \square*

Conjecture 0.4 (due to Vassiliev) *Filtration (0.2) “homotopically splits”, in other words, $\bar{\sigma}$ is “homotopy equivalent” to the wedge $\bigvee_{i=1}^{+\infty} (\bar{\sigma}_i / \bar{\sigma}_{i-1})$. \square*

This conjecture would imply the stabilization of our *main spectral sequence* in the first term over any commutative ring \mathbb{k} of coefficients.

Due to the Alexander duality, the filtration (0.2) induces the filtrations

$$H_{(0)}^*(\mathcal{K} \setminus \Sigma) \subset H_{(1)}^*(\mathcal{K} \setminus \Sigma) \subset H_{(2)}^*(\mathcal{K} \setminus \Sigma) \subset \dots, \quad (0.5)$$

$$H_*^{(0)}(\mathcal{K} \setminus \Sigma) \supset H_*^{(1)}(\mathcal{K} \setminus \Sigma) \supset H_*^{(2)}(\mathcal{K} \setminus \Sigma) \supset \dots \quad (0.6)$$

in respectively the cohomology and homology groups of the space of knots. For $d \geq 4$ the filtrations (0.5), (0.6) are finite for any dimension $*$. The Vassiliev spectral sequence in this case computes the graded quotient associated with the above filtrations.

In the most intriguing case $d = 3$ almost nothing is clear. For the dimension $* = 0$ the filtration (0.5) does not exhaust the whole cohomology of degree zero. The knot invariants obtained by this method are called the Vassiliev invariants, or invariants of finite type. One can define them in a more simple and geometrical way, see [V1]. The dual space to the graded quotient of the space of finite type knot invariants is the *bialgebra of chord diagrams*. The invariants and the bialgebra in question were intensively studied in the last decade, see [AF, BN1, CCL, ChD, GuMM, HV, K, L, P, S, Vai, Z]. The completeness conjecture for the Vassiliev knot invariants is the question about the convergence of the filtration (0.6) to zero for $d = 3$, $* = 0$. The realization theorem of M. Kontsevich [K] proves that the Vassiliev spectral sequence for $d = 3$, $* = 0$ also computes the corresponding associated quotient (for positive dimensions $*$ in the case $d = 3$ even this is not for sure) and does stabilize in the first term. The groups of the associated graded quotient to filtrations (0.5), (0.6) in the case $d = 3$, $* > 0$ are some quotient groups of the groups calculated by Vassiliev's main spectral sequence.

Let us mention that there is a natural way to construct real cohomology classes of the knot spaces by means of configuration space integrals, that generalizes in a non-trivial way the Vassiliev knot invariants obtained in three dimensions from the Chern-Simons perturbation theory, see [AF, BN2, CCL, GuMM, P]. May be this approach leads to a proof of Conjecture 0.3.

To compute the first term of the main spectral sequence, V. A. Vassiliev introduced an *auxiliary filtration* in the spaces $\sigma_i \setminus \sigma_{i-1}$, see [V1, V4]. The *auxiliary spectral sequence* associated to this filtration degenerates in the first term, because its first term (for any i) is concentrated at only one line. Therefore *the second term of the auxiliary spectral sequence is isomorphic to the first term of the main spectral sequence*. The term $E_0^{*,*}$ of the auxiliary spectral sequence together with its differential (of degree zero) is a direct sum of tensor products of complexes of connected graphs. The homology groups of the complex of connected graphs with m labelled vertices are concentrated in the dimension $(m - 2)$ only, and the only non-trivial group is isomorphic to $\mathbb{Z}^{(m-1)!}$, see [V3, V4]. This homology group has a nice description as the quotient by the 3-term relations of the space spanned by trees (with m labelled vertices), see [T2].

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In fact V. A. Vassiliev considered only cohomological case of the main and auxiliary spectral sequences, *i. e.*, the case corresponding to the homology of the discriminant and (by the Alexander duality) to the cohomology of the knot space $\mathcal{K} \setminus \Sigma$. This was because a convenient description only of the homology of complexes of connected graphs was known. It was noticed in [T3] that the cohomology of complexes of connected graphs (with m vertices) admits also a very nice description as the m -th component of the Lie algebras operad. This isomorphism comes from the following observation.

Let us consider the space $Inj(M, \mathbb{R}^d)$ of injective maps from a finite set M of cardinality m into \mathbb{R}^d , $d \geq 2$. This space can be viewed as a finite-dimensional analogue of the knot spaces. The corresponding discriminant (consisting of non-injective maps $M \rightarrow \mathbb{R}^d$) has also a simplicial resolution, whose filtration (analogous to (0.2)) does split homotopically, see [V2, V4]. The superior non-trivial term $\sigma_{m-1} \setminus \sigma_{m-2}$ of the filtration provides exactly

the complex of connected graphs with m vertices labelled by the elements of M . Its only non-trivial homology group is isomorphic by the Alexander duality to the cohomology group in the maximal degree of the space $\text{Inj}(M, \mathbb{R}^d)$. On the other hand the above space is homotopy equivalent to the m -th space of the little cubes operads, see Section 4. The homology operad of this topological operad is well known, see [Co] (and Section 4). For different d of the same parity this homology operad is the same up to a change of grading and for odd (resp. even) d it is the Poisson (resp. Gerstenhaber) algebras operad containing (in the maximal degree) the operad of Lie algebras.

An analogous periodicity takes place for the spaces of knots. The degree zero term of the main spectral sequence together with its differential (of degree zero) depends up to a change of grading on the parity of d only. Obviously the same is true for the whole auxiliary spectral sequence.

The above description of the cohomology of complexes of connected graphs allows one to describe easily the Vassiliev spectral sequence in the homological case. The main results of these computations are explained in Section 5 (see Theorems 5.5, 5.9, 5.10, 5.11). The proofs in full detail will be given in [T2], (see also the Russian version [T3]).

0.3

The paper is organized as follows.

In Sections 1, 2, 3 we give some preliminaries on linear graded operads. We give a short definition and examples (that will be useful for us) of linear graded operads (Section 1). We construct a graded Lie algebra structure on the space of any linear graded operad (Section 2). We define a Hochschild complex for any graded linear operad supplied with a morphism from the associative algebras operad to this operad (Section 3). The content of Sections 2 and 3 was borrowed (up to a slightly different definition of signs) from [GeV].

Section 4 is devoted to the May operad of little cubes and to its homology. As it was already mentioned the m -th component of this operad is a space homotopy equivalent to the space $\text{Inj}(\{1, 2, \dots, m\}, \mathbb{R}^d)$ of injective maps $\{1, 2, \dots, m\} \hookrightarrow \mathbb{R}^d$. In Section 4 we explain how the stratification in the discriminant set of non-injective maps $\{1, 2, \dots, m\} \rightarrow \mathbb{R}^d$ corresponds to a direct sum decomposition of the homology $H_*(\text{Inj}(\{1, 2, \dots, m\}, \mathbb{R}^d))$.

In Section 5 we describe a natural stratification in the discriminant set Σ of long singular ‘‘knots’’. This stratification provides a direct sum decomposition of the first term of the auxiliary spectral sequence. In the case of even d the first term of Vassiliev’s auxiliary spectral sequence is completely described by the following theorem:

Theorem 5.5. *The first term of Vassiliev’s homological auxiliary spectral sequence together with its first differential is isomorphic to the normalized Hochschild complex of the Batalin-Vilkovisky algebras operad. \square*

Unfortunately in the case of odd d it is not possible to describe the corresponding complex in terms of the Hochschild complex for some graded linear operad. A description of this complex is given in Section 8. Nevertheless the homology of the complex in question (*i. e.*, the first term of the main spectral sequence) over \mathbb{Q} can be defined in terms of the Hochschild homology of the Poisson algebras operad in the case of odd d (and of the Gerstenhaber algebras operad in the case of even d). A precise statement is given by Theorem 5.11.

In Section 5 we also introduce complexes homologically equivalent (for any commutative ring \mathbb{k} of coefficients) to the first term of the auxiliary spectral sequence. These

complexes simplify a lot the computations of the second term.

In Section 6 we explain how the bialgebra of chord diagrams arises in our construction. We formulate some problems concerning it.

Section 7 does not contain any new results and serves rather to explain one remark of M. Kontsevich. We study there the homology operads of some topological operads, that we call *operads of turning balls*. These homology operads make more clear the difference that we have in the cases of odd and even d .

In Section 8 we describe the first term of the auxiliary spectral sequence together with the degree 1 differential both for d even and odd. The corresponding complex is called *Complex of bracket star-diagrams*.

In Section 9 we construct a differential bialgebra structure on this complex. We conjecture that this differential bialgebra structure is compatible with the homology bialgebra structure of the space of long knots.

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1 Linear operads

The definition of many algebraic structures on a linear space (such as the commutative, associative, Lie algebra structures) consists of setting several polylinear operations (in these three cases, only one binary operation), that should satisfy some composition identities (in our example, associativity or Jacoby identity). Instead of doing this one can consider the spaces of all polylinear n -ary operations, for all $n \geq 0$, and the composition rules, that arize from the corresponding algebraic structure. The natural formalization of this object is given by the notion of operad.

Definition

Let \mathbb{k} be a commutative ring of coefficients. A *graded \mathbb{k} -linear operad* \mathcal{O} is a collection $\{\mathcal{O}(n), n \geq 0\}$ of graded \mathbb{k} -vector spaces equipped with the following set of data:

- (i) An action of the symmetric group S_n on $\mathcal{O}(n)$ for each $n \geq 2$.
- (ii) Linear maps (called *compositions*), preserving the grading,

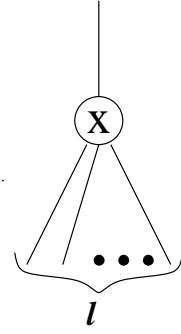
$$\gamma_{m_1, \dots, m_l} : \mathcal{O}(l) \otimes (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_l)) \rightarrow \mathcal{O}(m_1 + \cdots + m_l) \quad (1.1)$$

for all $m_1, \dots, m_l \geq 0$. We write $\mu(\nu_1, \dots, \nu_l)$ instead of $\gamma_{m_1, \dots, m_l}(\mu \otimes \nu_1 \cdots \otimes \nu_l)$.

- (iii) An element $id \in \mathcal{O}(1)$, called the *unit*, such that $id(\mu) = \mu(id, \dots, id) = \mu$ for any non-negative l and any $\mu \in \mathcal{O}(l)$.

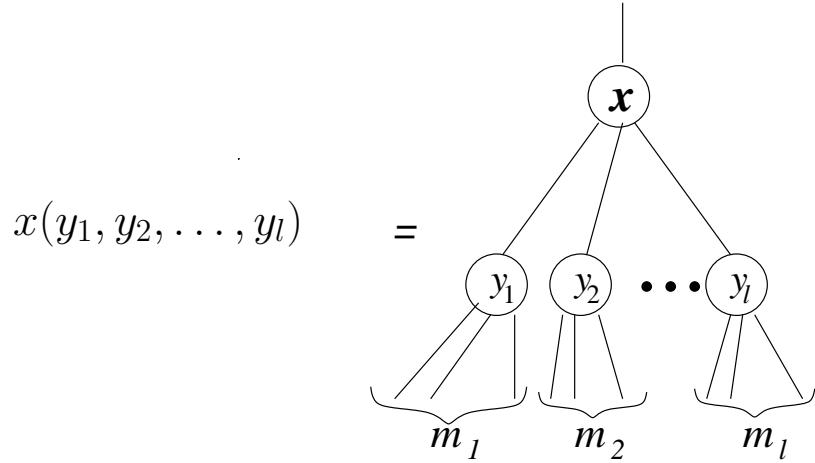
It is required that these data satisfy some conditions of associativity and equivariance with respect to the symmetric group actions, see [May, Chapter 1], [HS], [Lo].

One can consider any element $x \in \mathcal{O}(l)$ as something that has l inputs and 1 output:



(Figure 1.2)

The composition operation (1.1) of $x \in \mathcal{O}(l)$ with $y_1 \in \mathcal{O}(m_1), \dots, y_l \in \mathcal{O}(m_l)$ is the substitution of y_1, \dots, y_l into l inputs of x :



(Figure 1.3)

The resulting element has $m_1 + \dots + m_l$ inputs and 1 output, i. e., it belongs to $\mathcal{O}(m_1 + \dots + m_l)$.

Examples

We will give several well known examples of graded linear operads, see [GiK, Co, G].

a) Let V be a graded \mathbb{k} -vector space. We define the *endomorphism operad* $\mathcal{END}(V) := \{Hom(V^{\otimes n}, V), n \geq 0\}$. The unit element $id \in Hom(V, V)$ is put to be the identical map $V \rightarrow V$. The composition operations and the symmetric group actions are defined in the canonical way.

This operad is very important due to the following definition.

Definition 1.4 Let \mathcal{O} be a graded linear operad. By an *\mathcal{O} -algebra* (or *algebra over \mathcal{O}*) we call any couple (V, ρ) , where V is a graded vector space and ρ is a morphism $\rho : \mathcal{O} \rightarrow \mathcal{END}(V)$ of operads. \square

In other words, the theory of algebras over an operad \mathcal{O} is the representation theory of \mathcal{O} .

b) The *operad \mathcal{LIE} of Lie algebras*. The component $\mathcal{LIE}(0)$ of this operad is trivial. The n -th component $\mathcal{LIE}(n)$ is defined as the subspace of the free Lie algebra with generators x_1, x_2, \dots, x_n , that is spanned by the brackets containing each generator exactly once.

Example 1.5 For $n = 5$ one can take the bracket $[[[x_5, x_3], [x_1, x_2]], x_4]$ as an element of $\mathcal{LIE}(5)$. \square

Since we work in the category of graded vector spaces, we need to define a grading on each considered space. The grading of the spaces $\mathcal{LIE}(n)$, $n \geq 1$, is put to be zero. It is well known that these spaces are free \mathbb{k} -modules, $\mathcal{LIE}(n) \simeq \mathbb{k}^{(n-1)!}$. The S_n -action is defined by permutations of x_1, \dots, x_n .

Let $A(x_1, \dots, x_l)$, $B_1(x_1, \dots, x_{m_1})$, \dots , $B_l(x_1, \dots, x_{m_l})$ be brackets respectively from $\mathcal{LIE}(l)$, $\mathcal{LIE}(m_1)$, \dots , $\mathcal{LIE}(m_l)$. We define the composition operations (1.1) as follows.

$$A(B_1, \dots, B_l)(x_1, \dots, x_{m_1+\dots+m_l}) := A(B_1(x_1, \dots, x_{m_1}), B_2(x_{m_1+1}, \dots, x_{m_1+m_2}), \dots, \\ B_l(x_{m_1+\dots+m_{l-1}+1}, \dots, x_{m_1+\dots+m_l})). \quad (1.6)$$

The element $x_1 \in \mathcal{LIE}(1)$ is the unit element for this operad.

Note that a \mathcal{LIE} -algebra structure in the sense of Definition 1.4 is exactly the same as a (graded) Lie algebra structure in the usual sense. Indeed, the element $\rho([x_1, x_2]) \in Hom(V^{\otimes 2}, V)$ always defines a Lie bracket. The converse is also true. It is easy to see that the element $[x_1, x_2] \in \mathcal{LIE}(2)$ generates the operad \mathcal{LIE} , so if we put $\rho([x_1, x_2])$ equal to our Lie bracket, then we immediately obtain a map $\rho : \mathcal{LIE} \rightarrow \mathcal{END}(V)$ of the whole operad \mathcal{LIE} .

c) By analogy with the operad \mathcal{LIE} one defines the *operad \mathcal{COMM} (resp. \mathcal{ASSOC}) of commutative (resp. associative) algebras*. The space $\mathcal{COMM}(n)$ (resp. $\mathcal{ASSOC}(n)$), for $n \geq 1$, is put to be one-dimensional (resp. $n!$ -dimensional) free \mathbb{k} -module defined as the subspace of the free commutative (resp. associative) algebra with generators x_1, \dots, x_n , that is spanned by the monomials containing each generator exactly once. The S_n -actions and the composition operations are defined in the same way as for the operad \mathcal{LIE} . The element $x_1 \in \mathcal{COMM}(1)$ (resp. $\mathcal{ASSOC}(1)$) is the unit element *id*. There are two different ways to define the space $\mathcal{COMM}(0)$ (resp. $\mathcal{ASSOC}(0)$). We can put this space to be trivial or one-dimensional. In the first case we get the operad of commutative (resp. associative) algebras without unit, in the second case – the operad of commutative (resp. associative) algebras with unit. Below we will consider the second situation.

Remark 1.7 A commutative (resp. associative) algebra structure is exactly the same as a structure of \mathcal{COMM} -algebra (resp. \mathcal{ASSOC} -algebra) in the sense of Definition 1.4. \square

d) The *operads \mathcal{POISS} , \mathcal{GERST} , \mathcal{POISS}_d of Poisson, Gerstenhaber and d -Poisson algebras*. First of all, let us recall the definition of Poisson, Gerstenhaber and d -Poisson algebras.

Definition 1.8 A graded commutative algebra A is called a *d -Poisson algebra*, if it has a Lie bracket

$$[., .] : A \otimes A \rightarrow A$$

of degree $-d$. The bracket is supposed to be compatible with the multiplication. This means that for any elements $x, y, z \in A$

$$[x, yz] = [x, y]z + (-1)^{\tilde{y}(\tilde{x}-d)}y[x, z]. \quad \square \quad (1.9)$$

0-Poisson (resp. 1-Poisson) algebras are called simply *Poisson* (resp. *Gerstenhaber*) algebras. 1-Poisson algebras are called Gerstenhaber algebras in honor of Murray Gerstenhaber, who discovered this structure on the Hochschild cohomology of associative algebras, see [Ge] and also Sections 2 and 3.

Example 1.10 Let \mathfrak{g} be a graded Lie algebra with the bracket of degree $-d$. Then the symmetric (in the graded sense) algebra $S^*\mathfrak{g}$ has a natural structure of a d -Poisson algebra with the usual multiplication of a symmetric algebra and with the bracket defined by the following formula:

$$[A_1 \cdot A_2 \dots A_k, B_1 \cdot B_2 \dots B_l] = \sum_{i,j} (-1)^\epsilon A_1 \dots \widehat{A}_i \dots A_k \cdot [A_i, B_j] \cdot B_1 \dots \widehat{B}_j \dots B_l, \quad (1.11)$$

where $\epsilon = \tilde{A}_i(\sum_{i'=i+1}^k \tilde{A}_{i'}) + \tilde{B}_j(\sum_{j'=1}^{j-1} \tilde{B}_{j'}).$ \square

Remark 1.12 For a graded Lie algebra \mathfrak{g} with the bracket of degree 0 the d -tuple suspension $\mathfrak{g}[d]$ is also a graded Lie algebra with the bracket of degree $-d$. Thus, the space $S^*(\mathfrak{g}[d])$ is a d -Poisson algebra. \square

Note, that for commutative, associative, or Lie algebras the n -th component of the operad is defined as the space of all natural polylinear n -ary operations, that come from the corresponding algebra structure. Now, let us describe the spaces $\mathcal{POISS}_d(n)$ of all natural polylinear n -ary operations of d -Poisson algebras. Consider a free graded Lie algebra $Lie_d(x_1, \dots, x_n)$ with the bracket of degree $-d$ and with the generators x_1, \dots, x_n of degree zero, and consider (following Example 1.10) the d -Poisson algebra $Poiss_d(x_1, \dots, x_n) := S^*(Lie_d(x_1, \dots, x_n)).$ This is a free d -Poisson algebra. We will define the space $\mathcal{POISS}_d(n)$ as the subspace of $Poiss_d(x_1, \dots, x_n)$ spanned by the products (of brackets) containing each generator x_i exactly once. For instance for $n = 5$ we will take the product $[x_1, x_3] \cdot [[x_2, x_5]x_4]$ as an element of $\mathcal{POISS}_d(5).$ The unit element id is $x_1 \in \mathcal{POISS}_d(1).$ The symmetric group actions and the composition operations are defined analogously to the case of the operad $\mathcal{LIE}.$

The space $\mathcal{POISS}_d(n)$ can be decomposed into a direct sum with the summands numbered by partitions of the set $\{1, \dots, n\}:$

$$\mathcal{POISS}_d(n) = \bigoplus_A \mathcal{POISS}_d(A, n). \quad (1.13)$$

For a partition $A = \{\bar{A}_1, \dots, \bar{A}_{\#A}\}$ of the set $\{1, \dots, n\} = \coprod_{i=1}^{\#A} \bar{A}_i$ we define the space $\mathcal{POISS}_d(A, n) \subset \mathcal{POISS}_d(n)$ to be linearly spanned by products of $\#A$ brackets, such that the i -th bracket contains generators only from the set \bar{A}_i (thus, each generator from \bar{A}_i is presented exactly once in the i -th bracket).

Let $\bar{A}_1, \dots, \bar{A}_{\#A}$ be of cardinalities $a_1, \dots, a_{\#A}$ respectively, then

$$\mathcal{POISS}_d(A, n) \simeq \otimes_{i=1}^{\#A} \mathbb{k}^{(a_i-1)!}. \quad (1.14)$$

This implies that the space $\mathcal{POISS}_d(n)$ is isomorphic to $\mathbb{k}^{n!}$, and its Poincaré polynomial is $(1+t^{-d})(1+2t^{-d})\dots(1+(n-1)t^{-d})$.

e) The operad \mathcal{BV} (resp. \mathcal{BV}_d , d being odd) of Batalin-Vilkovisky (resp. d -Batalin-Vilkovisky) algebras.

Definition 1.15 A Gerstenhaber algebra (resp. d -Poisson algebra, for odd d) A is called a *Batalin-Vilkovisky algebra* (resp. *d -Batalin-Vilkovisky algebra*), if A is supplied with a linear map δ of degree -1 (resp. $-d$)

$$\delta : A \rightarrow A,$$

such that

- (i) $\delta^2 = 0$,
- (ii) $\delta(ab) = \delta(a)b + (-1)^{\tilde{a}}a\delta(b) + (-1)^{\tilde{a}}[a, b]$. \square

Note that (i) and (ii) imply

$$(iii) \delta([a, b]) = [\delta(a), b] + (-1)^{\tilde{a}+1}[a, \delta(b)].$$

Example 1.16 Let \mathfrak{g} be a graded Lie algebra with the bracket of degree zero. Then the exterior algebra $\Lambda^*\mathfrak{g} := S^*(\mathfrak{g}[1])$ is a Batalin-Vilkovisky algebra, where the structure of a Gerstenhaber algebra is from Remark 1.12; the operator δ is the standard differential on the chain-complex $\Lambda^*\mathfrak{g}$:

$$\delta(A_1 \wedge \dots \wedge A_k) = \sum_{i < j} (-1)^\epsilon [A_i, A_j] \wedge A_1 \dots \widehat{A_i} \dots \widehat{A_j} \dots \wedge A_k, \quad (1.17)$$

where $A_1, \dots, A_k \in \mathfrak{g}$, $\epsilon = \tilde{A}_i + (\tilde{A}_i + 1)(\tilde{A}_1 + \dots + \tilde{A}_{i-1} + i - 1) + (\tilde{A}_j + 1)(\tilde{A}_1 + \dots + \tilde{A}_{i-1} + \dots + \tilde{A}_{j-1} + j - 2)$. In the same way \mathfrak{g} defines the d -Batalin-Vilkovisky algebra $S^*(\mathfrak{g}[d])$ for any odd d . \square

Let us describe the n -th component of the corresponding operad, denoted by \mathcal{BV} and \mathcal{BV}_d (d being always odd). Since the spaces $\mathcal{BV}(n)$ and $\mathcal{BV}_d(n)$ are isomorphic (in the super-sense) up to a change of grading, we will consider now only the case $d = 1$. Obviously, the space $\mathcal{BV}(n)$ of all natural polylinear n -ary operations for such algebras contains $\mathcal{GERST}(n)$. Consider the symmetric algebra of the free graded Lie algebra $Lie_1(x_1, \dots, x_n, \delta(x_1), \dots, \delta(x_n))$ with the bracket of degree -1 and with the generators x_1, \dots, x_n of degree zero and the generators $\delta(x_1), \dots, \delta(x_n)$ of degree -1. This space has a structure of a Batalin-Vilkovisky algebra. (In fact it is a free \mathcal{BV} -algebra with generators x_1, \dots, x_n). In $S^*(Lie_1(x_1, \dots, x_n, \delta(x_1), \dots, \delta(x_n)))$ we will take the subspace $\mathcal{BV}(n)$ linearly spanned by all the products (of brackets), containing each index $i \in \{1, \dots, n\}$ exactly once. For instance $[\delta(x_1), x_3] \cdot [x_2, [\delta(x_4), \delta(x_5)]]$ belongs to $\mathcal{BV}(5)$. Due to relations (i), (ii), (iii) for δ , this subspace is exactly the space of all natural polylinear n -ary operations on Batalin-Vilkovisky algebras.

Analogously to (1.13) the space $\mathcal{BV}(n)$ can be decomposed into the direct sum:

$$\mathcal{BV}(n) = \bigoplus_{A, S} \mathcal{BV}(A, S, n), \quad (1.18)$$

where A is a partition of the set $\{1, \dots, n\}$, S is a subset of $\{1, \dots, n\}$ corresponding to indices i presented by $\delta(x_i)$ in products of brackets of $\mathcal{BV}(n)$.

Note that the space $\mathcal{BV}(n)$ is isomorphic to $\mathbb{k}^{2^n n!}$. The Poincaré polynomial of this graded space is $(1 + t^{-d})^{n+1}(1 + 2t^{-d})(1 + 3t^{-d}) \dots (1 + (n-1)t^{-d})$.

In the sequel we will use the following definition.

Definition 1.19 For a finite set M any pair (A, S) consisting of a partition A of M and of a subset $S \subset M$ will be called a *star-partition* of the set M . (Each point $i \in S$ will be called a *star*.) \square

Note that we have the following natural morphisms of operads:

$$\mathcal{ASSOC} \rightarrow \mathcal{COMM} \rightarrow \mathcal{POISS}_d \dashrightarrow \mathcal{BV}_d \quad (1.20)$$

The last arrow of (1.20) is defined only if d is odd.

2 Graded Lie algebra structure on graded linear operads

In this section we define a graded Lie algebra structure on an arbitrary graded linear operad. In the next section we use this structure and define the Hochschild complex for a graded linear operad equipped with a morphism from the operad \mathcal{ASSOC} to our operad. Both these constructions, which generalize the Hochschild cochain complex for associative algebras, were introduced in [GeV]. The only difference between the operations (2.1), (2.5), (3.5) given below and those of [GeV] is in signs. First of all, in the paper [GeV] M. Gerstenhaber and A. Voronov considered only linear (non-graded) operads, hence our case is more general. But even for the case of purely even gradings the signs are slightly different. The difference can be easily obtained by conjugation of the operations (2.1), (2.5), (3.5) by means of the linear operator that maps any element $x \in \mathcal{O}(n)$, $n \geq 0$, to $(-1)^{\frac{n(n-1)}{2}} x$.

Let $\mathcal{O} = \{\mathcal{O}(n), n \geq 0\}$ be a graded \mathbb{k} -linear operad. By abuse of the language the space $\bigoplus_{n \geq 0} \mathcal{O}(n)$ will be also denoted by \mathcal{O} . A tilde over an element will always designate its grading. For any element $x \in \mathcal{O}(n)$ we put $n_x := n - 1$. The numbers n and 1 here correspond to n inputs and to 1 output respectively.

Define a new grading $|.|$ on the space \mathcal{O} . For an element $x \in \mathcal{O}(n)$ we put $|x| := \tilde{x} + n_x = \tilde{x} + n - 1$. It turns out that \mathcal{O} is a graded Lie algebra with respect to the grading $|.|$. Note that the composition operations (1.1) respect this grading. Define the following collection of multilinear operations on the space \mathcal{O} .

$$x\{x_1, \dots, x_n\} := \sum (-1)^\epsilon x(id, \dots, id, x_1, id, \dots, id, x_n, id, \dots, id) \quad (2.1)$$

for $x, x_1, \dots, x_n \in \mathcal{O}$, where the summation runs over all possible substitutions of x_1, \dots, x_n into x in the prescribed order, $\epsilon := \sum_{p=1}^n n_{x_p} r_p + n_x \sum_{p=1}^n \tilde{x}_p + \sum_{p < q} n_{x_p} \tilde{x}_q$, r_p being the total number of inputs in x going after x_p . For instance, for $x \in \mathcal{O}(2)$ and arbitrary $x_1, x_2 \in \mathcal{O}$

$$x\{x_1, x_2\} = (-1)^{n_{x_1} + (\tilde{x}_1 + \tilde{x}_2) + n_{x_1} \tilde{x}_2} x(x_1, x_2).$$

We will also adopt the following convention:

$$x\{\} := x.$$

One can check immediately the following identities:

$$\begin{aligned}
& x\{x_1, \dots, x_m\}\{y_1, \dots, y_n\} = \\
& \sum_{0 \leq i_1 \leq j_1 \leq \dots \leq i_m \leq j_m \leq n} (-1)^\epsilon x\{y_1, \dots, y_{i_1}, x_1\{y_{i_1+1}, \dots, y_{j_1}\}, y_{j_1+1}, \dots, y_{i_m}, \\
& x_m\{y_{i_m+1}, \dots, y_{j_m}\}, y_{j_m+1}, \dots, y_n\}, \tag{2.2}
\end{aligned}$$

where $\epsilon = \sum_{p=1}^m (|x_p| \sum_{q=1}^{i_p} |y_q|)$. (These signs are the same as in [GeV]).

Define a bilinear operation (respecting the grading $|.|$) \circ on the space \mathcal{O} :

$$x \circ y := x\{y\}, \tag{2.3}$$

for $x, y \in \mathcal{O}$.

Definition 2.4 A graded vector space A with a bilinear operation

$$\circ : A \otimes A \rightarrow A$$

is called a *Pre-Lie algebra*, if for any $x, y, z \in A$ the following holds:

$$(x \circ y) \circ z - x \circ (y \circ z) = (-1)^{|y||z|}((x \circ z) \circ y - x \circ (z \circ y)) \square$$

Any graded Pre-Lie algebra A can be considered as a graded Lie algebra with the bracket

$$[x, y] := x \circ y - (-1)^{|x||y|}y \circ x. \tag{2.5}$$

The description of the operad of Pre-Lie algebras is given in [ChaL].

The following lemma is a corollary of the identity (2.2) applied to the case $m = n = 1$.

Lemma 2.6 *The operation (2.3) defines a graded Pre-Lie algebra structure on the space \mathcal{O} . \square*

In particular this lemma means that any graded linear operad \mathcal{O} can be considered as a graded Lie algebra with the bracket (2.5).

3 Hochschild complexes

Let $\mathcal{O} = \bigoplus_{n \geq 0} \mathcal{O}(n)$ be a graded linear operad equipped with a morphism

$$\Pi : \mathcal{ASSOC} \rightarrow \mathcal{O}$$

from the operad \mathcal{ASSOC} . This morphism defines the element $m = \Pi(m_2) \in \mathcal{O}(2)$, where the element $m_2 = x_1 x_2 \in \mathcal{ASSOC}(2)$ is the operation of multiplication. Note that the elements m_2, m are odd with respect to the new grading $|.|$ ($|m| = |m_2| = 1$) and $[m, m] = [\Pi(m_2), \Pi(m_2)] = 2\Pi(m_2 \circ m_2) = 0$. Thus \mathcal{O} becomes a differential graded Lie algebra with the differential ∂ :

$$\partial x := [m, x] = m \circ x - (-1)^{|x|}x \circ m, \tag{3.1}$$

for $x \in \mathcal{O}$.

The complex (\mathcal{O}, ∂) is called the *Hochschild complex* for the operad \mathcal{O} .

Example 3.2 If \mathcal{O} is the endomorphism operad $\mathcal{END}(A)$ of a vector space A , and we have a morphism

$$\Pi : \mathcal{ASSOC} \rightarrow \mathcal{END}(A),$$

that defines an associative algebra structure on A , then the corresponding complex $(\bigoplus_{n=0}^{+\infty} \text{Hom}(A^{\otimes n}, A), \partial)$ is the standard Hochschild cochain complex of the associative algebra A . \square

Example 3.3 Due to the morphisms (1.20) we have the Hochschild complexes $(\mathcal{ASSOC}, \partial)$, $(\mathcal{COMM}, \partial)$, $(\mathcal{POISS}_d, \partial)$, $(\mathcal{BV}_d, \partial)$. It can be shown that the complexes $(\mathcal{ASSOC}, \partial)$ and $(\mathcal{COMM}, \partial)$ are acyclic. \square

Define another grading

$$\deg := |.| + 1 \quad (3.4)$$

on the space \mathcal{O} . With respect to this grading the bracket $[., .]$ is homogeneous of degree -1 .

It is easy to see that the product $*$, defined as follows

$$x * y := (-1)^{|x|} m\{x, y\} = m(x, y), \quad (3.5)$$

for $x, y \in \mathcal{O}$, together with the differential ∂ defines a differential graded associative algebra structure on A with respect to the grading $\deg = |.| + 1$.

Theorem 3.6 [GeV] *The multiplication $*$ and the bracket $[., .]$ induce a Gerstenhaber (or what is the same 1-Poisson) algebra structure on the homology of the Hochschild complex (\mathcal{O}, ∂) . \square*

Proof of Theorem 3.6: The proof is deduced from the following homotopy formulas.

$$x * y - (-1)^{\deg(x)\deg(y)} y * x = (-1)^{\deg(x)} (\partial(x \circ y) - \partial x \circ y - (-1)^{\deg(x)-1} x \circ \partial y). \quad (3.7)$$

The above formula proves the graded commutativity of the multiplication $*$.

$$\begin{aligned} [x, y * z] - [x, y] * z - (-1)^{(\deg(x)-1)\deg(y)} y * [x, z] &= \\ &= (-1)^{\deg(x)+\deg(y)} (\partial(x\{y, z\}) - (\partial x)\{y, z\} - (-1)^{|x|} x\{\partial y, z\} - (-1)^{|x|+|y|} x\{y, \partial z\}). \end{aligned} \quad (3.8)$$

This formula proves the compatibility of the bracket with the multiplication. \square

4 The little cubes operad

Analogously to linear operads one can define topological operads, *i. e.*, collections $\{\mathcal{O}(n), n \geq 0\}$ of topological sets with

- (i) an S_n -action on each $\mathcal{O}(n)$;
- (ii) compositions

$$\gamma : \mathcal{O}(l) \times (\mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_l)) \rightarrow \mathcal{O}(m_1 + \cdots + m_l);$$

- (iii) a unit element $id \in \mathcal{O}(1)$.

We assume the same associativity and symmetric group equivariance requirements.

Evidently, the homology spaces $\{H_*(\mathcal{O}(n), \mathbb{k}), n \geq 0\}$ over any field \mathbb{k} form a graded \mathbb{k} -linear operad. The same is true for any commutative ring of coefficients, if the groups $H_*(\mathcal{O}(n), \mathbb{Z})$, $n \geq 0$, have no torsion.

Usually, when one considers the homology of a topological operad, one inverses the grading, supposing that the i -th homology group $H_i(\mathcal{O}(n), \mathbb{k})$ has the degree $-i$.

Historically one of the first examples of topological operads are the little cubes operads $\mathcal{LC}_d = \{\mathcal{LC}_d(n), n \geq 0\}$, $d \geq 1$, see [BV1], [BV2, Chapter 2], [May, Chapter 4]. Here $\mathcal{LC}_d(n)$ denotes the configuration space of n disjoint cubes labelled by $\{1, \dots, n\}$ in a unit cube. It is supposed that the faces of the cubes are parallel to the corresponding faces of the unit cube. The group S_n acts by permutations of the cubes in configurations. The element $id \in \mathcal{LC}_d(1)$ is the configuration of one cube coinciding with the unit cube. The composition operations (ii) are insertions of l configurations respectively of n_1, \dots, n_l cubes into the corresponding l cubes of a configuration of l cubes.

Theorem 4.1 [Co] *The homology groups of the operad \mathcal{LC}_d , $d \geq 1$, have no torsion and form the following graded linear operad*

- 1) if $d = 1$, $\{H_{-*}(\mathcal{LC}_1(n), \mathbb{k}), n \geq 0\}$ is isomorphic to the associative algebras operad \mathcal{ASSOC} ,
- 2) if $d \geq 2$, $\{H_{-*}(\mathcal{LC}_d(n), \mathbb{k}), n \geq 0\}$ is isomorphic to the $(d-1)$ -Poisson algebras operad $\mathcal{POISS}_{(d-1)}$. \square

Remark 4.2 The operad of associative algebras has a natural filtration compatible with the operad structure. The graded quotient associated with this filtration is the Poisson algebras operad \mathcal{POISS} . \square

We will not prove Theorem 4.1, however we will make some explanations.

First of all, note that the space $\mathcal{LC}_d(1)$ is contractible for any $d \geq 1$. The homology class of one point in this space corresponds to $id = x_1 \in \mathcal{POISS}_{(d-1)}(1)$. The space $\mathcal{LC}_d(2)$ is homotopy equivalent to the $(d-1)$ -dimensional sphere S^{d-1} . For $d \geq 2$ this gives us one operation of degree zero and one operation of degree $1-d$. The first operation corresponds to the multiplication $x_1 \cdot x_2 \in \mathcal{POISS}_{(d-1)}(2)$, the second one to the bracket $[x_1, x_2] \in \mathcal{POISS}_{(d-1)}(2)$.

Obviously, the space $\mathcal{LC}_d(n)$ is homotopy equivalent to the configuration space of collections of n distinct points in \mathbb{R}^d , i. e., to the space of injective maps $\{1, \dots, n\} \hookrightarrow \mathbb{R}^d$. The latter space is an open everywhere dense subset of the vector space \mathbb{R}^{nd} of all maps $\{1, \dots, n\} \rightarrow \mathbb{R}^d$. The complement (called the *discriminant*) $\Delta_d(n) \subset \mathbb{R}^{nd}$ is a union of $\frac{n(n-1)}{2}$ vector subspaces of codimension d . Each of these subspaces $L_{i,j}$ corresponds to a pair of distinct points $i, j \in \{1, \dots, n\}$ and consists of maps $\psi : \{1, \dots, n\} \rightarrow \mathbb{R}^d$, such that $\psi(i) = \psi(j)$.

The non-complete partitions of the set $\{1, \dots, n\}$ (by the *complete* partition we mean the partition into n singletons) are in one-to-one correspondence with the strata of the arrangement $\Delta_d(n) = \bigcup_{i < j} L_{i,j}$. To a partition we assign a vector subspace consisting of maps $\{1, \dots, n\} \rightarrow \mathbb{R}^d$ that glue the points of each set in the partition. Following the general theory of arrangements the reduced homology groups $\tilde{H}_*(\mathbb{R}^{nd} \setminus \Delta_d(n), \mathbb{k})$ can be decomposed into a direct sum, each summand being assigned to some stratum of the arrangement, see [GoMc, V2, V4, Ẑ]. In the case $d \geq 2$ this decomposition is canonical (in the case $d = 1$ it depends on the choice of a component of $\mathbb{R}^n \setminus \Delta_1(n)$). Let us assign to the complete partition the degree zero homology group $H_0(\mathbb{R}^{nd} \setminus \Delta_d(n), \mathbb{k}) \simeq \mathbb{k}$.

Proposition 4.3 *The above decomposition of the homology groups $H_{-*}(\mathbb{R}^{nd} \setminus \Delta_d(n), \mathbb{k})$, $d \geq 2$, coincides with the decomposition (1.13). \square*

5 The first term of the Vassiliev auxiliary spectral sequence

In the same way as the homology groups of the space of injective maps $\{1, \dots, n\} \hookrightarrow \mathbb{R}^d$ are decomposed into a direct sum by strata of the discriminant $\Delta_d(n)$ (see Section 4 — Proposition 4.3), the first term of the Vassiliev auxiliary spectral sequence is naturally decomposed into a direct sum in which the summands are numbered by equivalence classes of so called (A, b) -configurations defined below.

Let A be a non-ordered finite collection of natural numbers $A = (a_1, \dots, a_{\#A})$, any of which is not less than 2, and let b be a non-negative integer. Set $|A| := a_1 + \dots + a_{\#A}$. An (A, b) -configuration is a collection of $|A|$ distinct points in \mathbb{R}^1 separated into $\#A$ groups of cardinalities $a_1, \dots, a_{\#A}$, plus a collection of b distinct points in \mathbb{R}^1 (some of which can coincide with the above $|A|$ points). For short $(A, 0)$ -configuration are called simply A -configurations. A map $\phi : \mathbb{R}^1 \rightarrow \mathbb{R}^d$ respects an (A, b) -configuration, if it glues together all points inside any of its groups of cardinalities $a_1, \dots, a_{\#A}$, and the derivative ϕ' is equal to 0 at all the b last points of this configuration. For any (A, b) -configuration the set of maps respecting it is an affine subspace in \mathcal{K} of codimension $d(|A| - \#A + b)$; the number $|A| - \#A + b$ is called the *complexity* of the configuration. Two (A, b) -configurations are called *equivalent* if they can be transformed into one another by an orientation-preserving homeomorphism $\mathbb{R}^1 \rightarrow \mathbb{R}^1$.

Consider any (A, b) -configuration J of complexity i and with j geometrically distinct points in \mathbb{R}^1 . The stratum consisting of all mappings $\mathbb{R}^1 \rightarrow \mathbb{R}^d$ respecting at least one (A, b) -configuration J' equivalent to J , can be parameterized by the space $S(J)$ of affine fiber bundle, whose base space is the space E^j of (A, b) -configurations J' equivalent to J , and the fiber over J' is the affine space $\mathbb{R}^{\omega-di}$ of maps respecting J' . Note that E^j is contractible being an open cell of dimension j . Therefore this fiber bundle can be trivialized:

$$S(J) \simeq E^j \times \mathbb{R}^{\omega-di}. \quad (5.1)$$

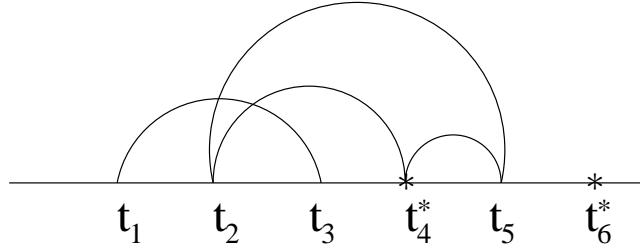
Remark 5.2 The corresponding stratum is not homeomorphic to $S(J)$: one map $\mathbb{R}^1 \rightarrow \mathbb{R}^d$ may respect two different (A, b) -configurations equivalent to J , and therefore the stratum has self-intersections. \square

The auxiliary spectral sequence computing the homology groups of the term $\sigma_i \setminus \sigma_{i-1}$ in our filtration (0.2) uses those and only those (A, b) -configurations, that are of complexity i . The geometrical meaning of the first differential in the auxiliary spectral sequence is in how the strata (or rather the base spaces of the corresponding affine bundles (5.1)) corresponding to equivalence classes of (A, b) -configurations of complexity i bound to each other.

Let first d be even. For any equivalence class of (A, b) -configurations of complexity i and with j geometrically distinct points we will assign a star-partition (\bar{A}, S) of the set $\{1, \dots, j\}$ (see Definition 1.19) and therefore a subspace $\mathcal{BV}_{(d-1)}(\bar{A}, S, j)$ of the operad $\mathcal{BV}_{(d-1)}$ (see Decomposition 1.18), where the partition \bar{A} of the set $\{1, \dots, j\}$ and the subset $S \subset \{1, \dots, j\}$ are defined below.

Definition 5.3 A *minimal component* of an (A, b) -configuration is either one of its b points, which does not coincide with none of the $|A|$ points, or one of the $\#A$ groups of points with all the stars contained there. \square

For instance, the (A, b) -configuration on the Figure 5.4 has 3 minimal components consisting respectively of the following groups of points: 1) t_1, t_3 ; 2) t_2, t_4^*, t_5 ; 3) t_6^* .



(Figure 5.4)

To any (A, b) -configuration with j geometrically distinct points we can assign a star-partition of the set $\{1, \dots, j\}$. Let $t_1 < t_2 < \dots < t_j$ be the points in \mathbb{R}^1 of our (A, b) -configuration (b of them are marked by stars). The set of minimal components of this (A, b) -configuration defines a partition \bar{A} of the set $\{1, \dots, j\}$. We also have the subset $S \subset \{1, \dots, j\}$ (of cardinality b) of indices, corresponding to the points marked by stars.

Note that different equivalence classes of (A, b) -configurations correspond to different star-partitions. But the correspondence is far from being bijective. The star-partitions not corresponding to (A, b) -configurations are those and only those, which contain singletons not marked by a star.

Theorem 5.5 see [T2]. *The first term of the Vassiliev auxiliary spectral sequence for even d together with its first differential is isomorphic to the subcomplex of the Hochschild complex $(\mathcal{BV}_{(d-1)}, \partial)$, linearly spanned by the summands of the decomposition (1.18), corresponding to star-partitions, which don't contain singletons not marked by a star. The grading corresponding to the homology of the knot space $\mathcal{K} \setminus \Sigma$ is minus the grading “deg” defined by (3.4). \square*

In particular, the theorem claims that the subspace in $\mathcal{BV}_{(d-1)}$ spanned by the summands in question is invariant with respect to the differential ∂ .

Consider the decomposition of the space $\mathcal{BV}_{(d-1)} = \bigoplus_{i \geq 0} E_i$, where E_i is the sum over all star-partitions having exactly i singletons not marked by a star. The filtration $F_0 \supset F_1 \supset F_2 \dots$, with $F_i := \bigoplus_{j \geq i} E_j$, is compatible with the differential ∂ . Note, that the complex (E_0, ∂) is exactly the complex from Theorem 5.5.

Proposition 5.6 [T2] *The Hochschild complex $(\mathcal{BV}_{(d-1)}, \partial)$ (d is even) is a direct sum of the complexes (E_0, ∂) and (F_1, ∂) . The first complex (E_0, ∂) is homology (and even homotopy) equivalent to $(\mathcal{BV}_{(d-1)}, \partial)$; the second one (F_1, ∂) is acyclic (and even contractible). \square*

An analogous statement holds for the complexes $(\mathcal{POISS}_{(d-1)}, \partial)$, d being any integer number. The subcomplexes spanned by the summands of the decomposition (1.13) corresponding to partitions non-containing singletons will be called the *normalized Hochschild*

complexes and denoted by $(\mathcal{POISS}_{(d-1)}^{\text{Norm}}, \partial)$. The *normalized Hochschild complexes* (E_0, ∂) for the operads $\mathcal{BV}_{(d-1)}$, d being even, will be denoted also by $(\mathcal{BV}_{(d-1)}^{\text{Norm}}, \partial)$.

Corollary 5.7 The first term of the main spectral sequence (for even d and for any commutative ring \mathbb{k} of coefficients) is isomorphic to the Hochschild homology (with inverted grading) of the Batalin-Vilkovisky operad $\mathcal{BV}_{(d-1)}$. \square

Remark 5.8 (due to M. Kontsevich) The operator $\delta \in \mathcal{BV}_{(d-1)}(1)$ has a natural geometrical interpretation as the Euler class $\delta_{d-1}^E \in H_{d-1}(SO(d))$ of the special orthogonal group $SO(d)$, see Section 7. \square

If d is odd, then the first term of the auxiliary spectral sequence is very similar to the normalized Hochschild complex $(\mathcal{BV}^{\text{Norm}}, \partial)$, but it does not correspond to any operad. A description of the obtained complex, which I called the *complex of bracket star-diagrams*, see in Section 8.

In [T2] the following theorem is proved. For even d this is an immediate corollary of Theorem 5.5.

Theorem 5.9 For any $d \geq 3$ the subspace of the first term of the auxiliary spectral sequence linearly spanned only by the summands corresponding to A -configurations forms a subcomplex isomorphic to the normalized Hochschild complex $(\mathcal{POISS}_{(d-1)}^{\text{Norm}}, \partial)$ with inverted grading. \square

There is a very nice way to simplify the computations of the second term of the auxiliary spectral sequence. This construction works both for d even and odd. Consider the normalized Hochschild complex $(\mathcal{POISS}_{(d-1)}^{\text{Norm}}, \partial)$ and consider the quotient of the space of this complex by the “neighboring commutativity relations” — in other words, for any $i = 1, 2, \dots$ we set x_i and x_{i+1} to commute. For example the element $[x_1, x_3] \cdot [x_2, [x_4, x_5]] \in \mathcal{POISS}_{(d-1)}(5)$ is equal to zero modulo these relations, because it contains $[x_4, x_5]$. The space of relations is invariant with respect to the differential, thus the quotient space has the structure of a quotient complex. Denote this quotient complex by $(\mathcal{POISS}_{(d-1)}^{\text{zero}}, \partial)$.

Theorem 5.10 [T2, T3] For any commutative ring \mathbb{k} of coefficients the space $\mathcal{POISS}_{(d-1)}^{\text{zero}}$ is a free \mathbb{k} -module. The homology space of the complex $(\mathcal{POISS}_{(d-1)}^{\text{zero}}, \partial)$ is isomorphic to the first term of the main spectral sequence (=to the second term of the auxiliary spectral sequence). \square

Idea of the proof: Consider the filtration in the first term (of the auxiliary spectral sequence) by the number of minimal components of corresponding (A, b) -configurations. The associated spectral sequence degenerates in the second term, because its first term is concentrated on the only line corresponding to A -configurations. \square

Before describing the relation of the Hochschild homology space of $\mathcal{POISS}_{(d-1)}$ defined over \mathbb{Q} with the first term of the main spectral sequence also defined over \mathbb{Q} (see Theorem 5.11) we will give some explanations.

In Section 8 a structure of a differential graded cocommutative bialgebra on the first term of the auxiliary spectral sequence is defined. According to Theorems 5.5 and 5.9 such a structure is defined also on the normalized Hochschild complexes $(\mathcal{POISS}^{\text{Norm}}, \partial)$, $(\mathcal{GERST}^{\text{Norm}}, \partial)$, $(\mathcal{BV}^{\text{Norm}}, \partial)$. The neighboring commutativity relations respect multiplication and comultiplication (but not the bracket (2.5)), so the

complexes $(\mathcal{POISS}_{(d-1)}^{\text{zero}}, \partial)$ are differential bialgebras. A motivation of the existence of such a structure is that the space of long knots is an H -space (has a homotopy associative multiplication), see [T1, T2, T3]; therefore its homology and cohomology spaces over any field \mathbb{k} are mutually dual graded respectively cocommutative and commutative bialgebras. For $\mathbb{k} = \mathbb{Q}$ these (co)homology bialgebras are bicommutative, see [T2], [T3]. This fact supports Conjecture 0.3, because applying Theorem 3.6 to the operads of Poisson, Gerstenhaber or Batalin-Vilkovisky algebras over any field \mathbb{k} , we obtain that their Hochschild homology bialgebras are bicommutative. It follows from the Milnor theorem, see [MM], that for $\mathbb{k} = \mathbb{Q}$ these bialgebras are polynomial. The space of primitive elements is the space of their generators. In fact, the bracket 2.5 for these operads preserves the spaces of primitive elements. We conclude that as Gerstenhaber algebras the Hochschild homology spaces of these operads are symmetric algebras of the Lie algebras of primitive elements (see Example 1.10 with $d = 1$).

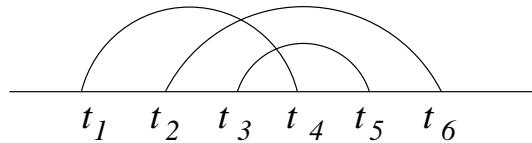
Theorem 5.11 [T2, T3] *As a graded bialgebra the first term of the main spectral sequence over \mathbb{Q} is isomorphic to the Hochschild homology bialgebra (with the inversed grading) of the \mathbb{Q} -linear operad \mathcal{POISS}_d factorized*

- 1) for even d : by one odd primitive generator $[x_1, x_2]$ (of degree $3 - d$);
- 2) for odd d : by one even primitive generator $[x_1, x_2]$ (of degree $3 - d$) and one odd primitive generator $[[x_1, x_3], x_2]$ (of degree $5 - 2d$). \square

6 The bialgebra of chord diagrams

The *bialgebra of chord diagrams*, the dual to the associated quotient bialgebra of Vassiliev knot invariants, was intensively studied in the last decade. In this section we give an interpretation of the bialgebra of chord diagrams as a part of the Hochschild homology algebra of the Poisson algebras operad.

Consider the normalized Hochschild complex $(\mathcal{POISS}_{(d-1)}^{\text{Norm}}, \partial)$. In this complex one can define a bigrading by the complexity i and by the number $j = |A|$ of geometrically distinct points of the corresponding A -configurations. The differential ∂ is of bidegree $(0,1)$. If the first grading i is fixed, then the number j varies from $i + 1$ to $2i$. So any element of the bigrading $(i, 2i)$ belongs to the kernel of ∂ . The case $j = 2i$ corresponds to the minimal possible dimension of non-trivial homology classes of the space of long knots for the complexity i fixed. For example, if $d = 3$, then this dimension is equal to $(d - 1)i - j = 2i - j = 0$. The part of the Hochschild homology groups, that lies in the bigradings $(i, 2i)$, $i \geq 0$, will be called the *bialgebra of chord diagrams* if d is odd, and the *bialgebra of chord superdiagrams* if d is even. Any product of brackets in $\mathcal{POISS}_{(d-1)}^{\text{Norm}}(2i) \subset \text{Poiss}_{(d-1)}(x_1, \dots, x_{2i}) = S^* \text{Lie}_{(d-1)}(x_1, \dots, x_{2i})$ of bidegree $(i, 2i)$ is the product of i brackets, each of which contains exactly 2 generators. Thus, any such product of brackets can be depicted as $2i$ points on the line \mathbb{R}^1 , that are decomposed into i pairs and connected by a chord inside each pair. For example, $[x_3, x_5] \cdot [x_4, x_1] \cdot [x_2, x_6]$ is assigned to the diagram

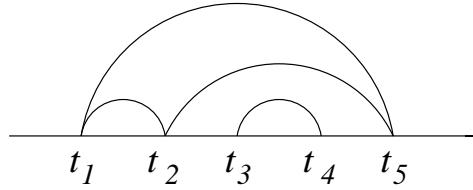


(Figure 6.1)

The so called *4-term relations* arise as the differential ∂ of products of brackets, in which all brackets except one are “chords”, and the only non-chord is a bracket on three elements. In other words, these products of brackets correspond to A -configurations, with $A = (3, 2, 2, \dots, 2)$.

Remark 6.2 Sometimes one considers the bialgebra of chord diagrams factorized not only by 4-term relations, but also by 1-term relations. The latter relations arise (when we consider the whole first term of the auxiliary spectral sequence) as the differential of diagrams corresponding to (A, b) -configurations, with $A = (2, \dots, 2)$, $b = 1$, and the only star does not coincide with none of $|A|$ points. \square

In the case of odd d we need to take into account orientations of chords (since $[x_{i_1}, x_{i_2}] = -[x_{i_2}, x_{i_1}]$). This definition does not coincide with the standard one, where these orientations are not important. This discordance of definitions can be easily eliminated. Consider the Hochschild complex $(\mathcal{POISS}_{(d-1)}, \partial) = (\bigoplus_{n \geq 0} \mathcal{POISS}_{(d-1)}(n), \partial)$, and replace each space $\mathcal{POISS}_{(d-1)}(n)$ by its tensor product with the one-dimensional sign representation *sign* of the symmetric group S_n . This can be done in the following way. We take a free $(d-1)$ -Poisson algebra $Poiss_{(d-1)}(x'_1, \dots, x'_n) = S^* Lie(x'_1, \dots, x'_n)$ with generators x'_1, \dots, x'_n of degree one instead of the analogous algebra $Poiss_{(d-1)}(x_1, \dots, x_n) \supset \mathcal{POISS}_{(d-1)}(n)$ with generators x_1, \dots, x_n of degree zero. Afterwards we consider the subspace $\mathcal{POISS}'_{(d-1)}(n) \subset Poiss_{(d-1)}(x'_1, \dots, x'_n)$ spanned by products of brackets containing each generator x'_i exactly once. Obviously, the S_n -module $\mathcal{POISS}'_{(d-1)}(n)$ is isomorphic to $\mathcal{POISS}_{(d-1)}(n) \otimes \text{sign}$. Defining properly the differential, see Section 8, we obtain another version of the Hochschild complex for the operads $\mathcal{POISS}_{(d-1)}$. This new version is interpreted in the geometry of the discriminant Σ as introducing an orientation of the spaces E^j of the strata (5.1) not according to the usual order $t_1 < t_2 < \dots < t_j$ of the points on the line \mathbb{R}^1 , but according to the order, that was in our product of brackets. For instance, for the element $[x_3, x_4] \cdot [x_5, [x_2, x_1]]$ the orientation of the corresponding space $E^5 = \{t_1 < t_2 < t_3 < t_4 < t_5\}$ of $(3, 2)$ -configurations equivalent to configuration (6.3) is according to the order $(t_3, t_4, t_5, t_2, t_1)$.



(Figure 6.3)

An advantage of the new Hochschild complexes (for operads $\mathcal{POISS}_{(d-1)}$) is a simpler rule of signs in the definition of the differential, see Section 8.

Let \mathcal{O} be a graded linear operad, equipped with a morphism from the operad \mathcal{ASSOC} ,

then any \mathcal{O} -algebra A is an associative algebra because of the following morphisms:

$$\mathcal{ASSOC} \rightarrow \mathcal{O} \rightarrow \mathcal{END}(A).$$

On the other hand, the map $\mathcal{O} \rightarrow \mathcal{END}(A)$ defines a morphism of Hochschild complexes:

$$(\mathcal{O}, \partial) \rightarrow (\mathcal{END}(A), \partial).$$

Therefore the classes in the Hochschild homology of \mathcal{O} can be considered as characteristic classes of the Hochschild cohomology of \mathcal{O} -algebras (considered as associative algebras). An interesting question is whether all the classes in the Hochschild homology of \mathcal{BV} , \mathcal{POISS} , \mathcal{GERST} , $\mathcal{BV}_{(d-1)}$, $\mathcal{POISS}_{(d-1)}$ have a non-trivial realization as characteristic classes.

Consider Example 1.10, where we take $S^*(\mathfrak{g}[d-1])$ as a $(d-1)$ -Poisson algebra. The Hochschild homology space of a polynomial algebra is well known. It is the space of polynomial polyvector fields on the space of generators. In our case the space of generators is $\mathfrak{g}[d-1]$. According to the grading rule, we get that the Hochschild homology space of $\mathcal{POISS}_{(d-1)}$ of bigrading (i, j) is mapped to the space of homogenous degree i j -polyvector fields, *i. e.*, of expressions of the form

$$\sum_{\substack{q_1, \dots, q_j \\ p_1, \dots, p_i}} A_{p_1 \dots p_i}^{q_1 \dots q_j} x^{p_1} \dots x^{p_i} \frac{\partial}{\partial x^{q_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{q_j}}. \quad (6.4)$$

If d is odd (resp. even), the tensor $A_{p_1 \dots p_i}^{q_1 \dots q_j}$ is symmetric (resp. antisymmetric) with respect to the indices p_1, \dots, p_i and antisymmetric (resp. symmetric) with respect to the indices q_1, \dots, q_j .

Problem 6.5. Find explicitly $A_{p_1 \dots p_i}^{q_1 \dots q_j}$ via the bracket on \mathfrak{g} , for example, in the case $j = 2i$ of chord diagrams. \square

The answer to Problem 6.5 can be related with the invariant tensors, defined for Casimir Lie algebras by chord diagrams, see [BN1, HV, Vai].

Also a problem which looks interesting is to find the Gerstenhaber subalgebra in the Hochschild homology of the operads \mathcal{POISS} , \mathcal{GERST} generated by the space of chord (super)diagrams. For instance, in the case of the operad \mathcal{GERST} there is an element of bigrading $(3, 5)$, that cannot be obtained from chord diagrams by means of the multiplication and the bracket (2.5), see [T1, T2]. In the case of the operad \mathcal{POISS} we do not know such an example.

7 Operads of turning balls

In this section we clarify the difference between the cases of odd and even d and give a geometrical interpretation of the operad $\mathcal{BV}_{(d-1)}$ (for d even). In particular we explain Remark 5.8 (due to M. Kontsevich). Theorems 7.1, 7.2 given below are classical, see [G], [B].

For any $d \geq 2$ let us introduce the topological operad $\mathcal{TB} = \{\mathcal{TB}_d(n), n \geq 0\}$ of turning balls. The space $\mathcal{TB}_d(n)$ is put to be the configuration space of n mappings of the unit ball $B^d \subset \mathbb{R}^d$ into itself. These n mappings are supposed

- 1) to be injective, preserving the orientation and the ratio of distances;

2) to have disjoint images.

The space $\mathcal{TB}_d(n)$ can be evidently identified with the direct product of the n -th power $(SO(d))^n$ of the special orthogonal group with the configuration space of n disjoint balls in the unit ball B^d . The last space is homotopy equivalent to the n -th component $\mathcal{LC}_d(n)$ of the little cubes operad. The composition operations, the symmetric group actions and the identity element are defined analogously to the case of the little cubes operad.

Theorem 7.1 [G] *The homology $\{H_{-*}(\mathcal{TB}_2(n), \mathbb{Z}), n \geq 0\}$ of the turning discs operad (balls of dimension 2) is the Batalin-Vilkovisky algebras operad \mathcal{BV} . \square*

Let us describe the operad $\{H_{-*}(\mathcal{TB}_d(n), \mathbb{Q}), n \geq 0\}$ for any $d \geq 2$. Note, that the space of unary operations is the homology algebra $H_{-*}(SO(d), \mathbb{Q})$.

Theorem 7.2 [B] *The homology bialgebra $H_*(SO(d), \mathbb{Q})$ is the exterior algebra on the following primitive generators:*

- 1) *case $d = 2k + 1$: generators $\delta_3, \delta_7, \dots, \delta_{4k-1}$ of degree $3, 7, \dots, 4k - 1$ respectively;*
- 2) *case $d = 2k$: generators $\delta_3, \delta_7, \dots, \delta_{4k-5}, \delta_{2k-1}^E$ of degree $3, 7, \dots, 4k - 5$ and $2k - 1$ respectively.* \square

The generators $\delta_{4i-1} \in H_{4i-1}(SO(d), \mathbb{Q})$, $d \geq 2i + 1$, are called the *Pontriagin classes*, the generator $\delta_{2k-1}^E \in H_{2k-1}(SO(2k), \mathbb{Q})$ is called the *Euler class*. The Pontriagin classes (and the subalgebra generated by them) lie in the kernel of the map in homology $H_*(SO(d), \mathbb{Q}) \rightarrow H_*(S^{d-1}, \mathbb{Q})$ induced by the natural projection

$$SO(d) \xrightarrow{SO(d-1)} S^{d-1}.$$

The Euler class is sent to the canonical class of dimension $d - 1$ in the homology of the sphere S^{d-1} .

Now we are ready to describe the operads $\{H_{-*}(\mathcal{TB}_d(n), \mathbb{Q}), n \geq 0\}$, $d \geq 2$. We will say which objects are algebras over these operads.

Theorem 7.3 [T2] *Algebras over the operad $\{H_{-*}(\mathcal{TB}_d(n), \mathbb{Q}), n \geq 0\}$ are*

- 1) *for even d : $(d - 1)$ -Batalin-Vilkovisky algebras, where the operator δ is δ_{d-1}^E ;*
- 2) *for odd d : $(d - 1)$ -Poisson algebras.*

Furthermore these algebras are supposed to have $[\frac{d-1}{2}]$ (where “ $[\cdot]$ ” denotes the integral part) mutually super-comuting differentials $\delta_3, \delta_7, \dots, \delta_{4[\frac{d-1}{2}]-1}$ (of degree $-3, -7, \dots, 1 - 4[\frac{d-1}{2}]$) of the Batalin-Vilkovisky (resp. Poisson) algebra structure. This means, that for any elements a, b of the algebra and $1 \leq i \leq [\frac{d-1}{2}]$, one has

- (i) $\delta_{4i-1}([a, b]) = [\delta_{4i-1}a, b] + (-1)^{\tilde{a}+d-1}[a, \delta_{4i-1}b];$
- (ii) $\delta_{4i-1}(a \cdot b) = (\delta_{4i-1}a) \cdot b + (-1)^{\tilde{a}}a \cdot (\delta_{4i-1}b);$
- (iii) *only for even d :* $\delta_{4i-1}\delta_{d-1}^Ea = -\delta_{d-1}^E\delta_{4i-1}a.$ \square

8 Complexes of bracket star-diagrams

In this section we describe the first term of the auxiliary spectral sequence together with its first differential. The corresponding complex will be called *complex of bracket star-diagrams*. In the case of even d this complex is isomorphic to the normalized Hochschild complex of the Batalin-Vilkovisky algebras operad $(\mathcal{BV}^{Norm}, \partial)$.

8.1 Case of odd d

Let us fix an (A, b) -configuration J . Let t_α , $\alpha \in \alpha$ (resp. t_β^* , $\beta \in \beta$) be the points of J on the line \mathbb{R}^1 that do not have stars (resp. that do have stars). Consider the free Lie super-algebra with the even bracket and with odd generators x_{t_α} , $\alpha \in \alpha$, $x_{t_\beta^*}$, $\beta \in \beta$. We will take the symmetric algebra (in the super-sens) of the space of this Lie super-algebra. In the obtained space we will consider the subspace $\mathcal{BSD}(J)$ spanned by the (products of brackets), where each minimal component of J is presented by one bracket, containing only generators indexed by the points of this minimal component and containing each such generator exactly once.

Such products of brackets will be called *bracket star-diagrams*.

Example 8.1 The space $\mathcal{BSD}(J)$ of the bracket star-diagrams corresponding to the (A, b) -configuration J of the Figure 5.4 is two-dimensional. The diagrams

$$[x_{t_1}, x_{t_3}] \cdot [[x_{t_2}, x_{t_4^*}], x_{t_5}] \cdot x_{t_6^*}, [x_{t_1}, x_{t_3}] \cdot [[x_{t_2}, x_{t_5}] x_{t_4^*}] \cdot x_{t_6^*}$$

form a basis in this space. \square

If two bracket star-diagrams can be transformed into one another by an orientation preserving homeomorphism $\mathbb{R}^1 \rightarrow \mathbb{R}^1$, then they are set to be equal. For any equivalence class \mathbf{J} of (A, b) -configurations we define the space $\mathcal{BSD}(\mathbf{J})$ as the space $\mathcal{BSD}(J)$, where J is any element of \mathbf{J} .

The *space of bracket star-diagrams* is defined as the direct sum of the spaces $\mathcal{BSD}(\mathbf{J})$ over all equivalence classes \mathbf{J} of (A, b) -configurations.

The complexity i and the number j of geometrically distinct points of the corresponding (A, b) -configurations define the bigrading (i, j) on the space of bracket star-diagrams. Remind that the complex of bracket star-diagrams (both for d odd and even) is supposed to compute the first term $E_{p,q}^1$ of the main spectral sequence, whose (p, q) coordinates are expressed as follows:

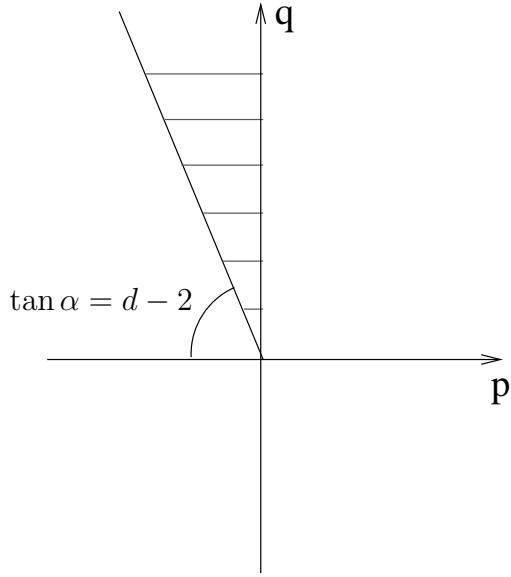
$$p = -i,$$

$$q = id - j.$$

The corresponding homology degree of the space of knots $\mathcal{K} \setminus \Sigma$ is

$$p + q = i(d - 1) - j. \tag{8.2}$$

Note that the first term of the main spectral sequence is non-trivial only in the second quadrant $p \leq 0$. The inequality $j \leq 2i$ for (A, b) -configurations provides the condition $q \geq (2 - d)p$.



(Figure 8.3)

The bialgebra of chord (super)diagrams occupies the diagonal $q = (2 - d)p$.

To describe the differential ∂ on the space of bracket star-diagrams we will need some complementary definitions and notations.

Definition 8.4 Let us permit to (A, b) -configurations, with $A = (a_1, \dots, a_{\#A})$, to have $a_i = 1$; we demand also that one-element sets should never coincide with stars. These (A, b) -configurations will be called *generalized (A, b) -configurations*. The generalized (A, b) -configurations that are not (A, b) -configurations in the usual sens (that have $a_i = 1$) will be called *special generalized (A, b) -configurations*. \square

Analogously we define the *space of (special) generalized bracket star-diagrams*.

Example 8.5 $[x_{t_1}, x_{t_3}] \cdot x_{t_2}$ is a special generalized star-diagram. \square

Note that the space of generalized bracket star-diagrams is the direct sum of the space of bracket star-diagrams with the space of special generalized bracket star-diagrams.

Definition 8.6 We say that a (generalized) (A, b) -configuration J can be *inserted* in a point $t_0^{(*)}$ of another (generalized) (A', b') -configuration J' , if J does not have common points with J' except possibly the point $t_0^{(*)}$. \square

Definition 8.7 We say that a (generalized) bracket star-diagram can be *inserted* in the point $t_0^{(*)}$ of another (generalized) bracket star-diagram, if it is the case for their (A, b) -configurations. \square

Let A and B be two generalized bracket star-diagrams, such that A can be inserted in the point t_0 (or t_0^*) of B , define the element $B|_{x_{t_0}=A}$ (resp. $B|_{x_{t_0^*}=A}$) of the space of generalized bracket star-diagrams. Up to a sign $B|_{x_{t_0^*}=A}$ is defined by replacing $x_{t_0^*}$ (in

the diagram B) for A . The sign is defined as $(-1)^{(\tilde{A}-1) \times n}$, where \tilde{A} is the parity of A (the parity of the number of geometrically distinct points), n is the number of generators of the form $x_{t_\alpha}, x_{t_\beta^*}$ before $x_{t_0^{(*)}}$ in B . In other words : we put the bracket containing $x_{t_0^{(*)}}$ on the first place, then by means of antisymmetry relations we put $x_{t_0^{(*)}}$ on the first place in the bracket (and therefore in the diagram); we replace $x_{t_0^{(*)}}$ for A ; and we do all these manipulations in the inverse order. It is easy to see that these two definitions give the same sign.

Example 8.8

$$[x_{t_2} x_{t_3^*}] \cdot [x_{t_1^*} x_{t_0}] \Big|_{x_{t_0} = [x_{t_4} x_{t_5}] \cdot x_{t_6^*}} = (-1)^{(3-1) \cdot 3} [x_{t_2} x_{t_3^*}] \cdot [x_{t_1^*}, [x_{t_4} x_{t_5}] \cdot x_{t_6^*}]. \square$$

Note that if A has more than 1 minimal components, then the element $B|_{x_{t_0^{(*)}}=A}$ contains multiplications inside brackets. Therefore it is no more a (generalized) bracket star-diagram. To express this element as a sum of (generalized) bracket star-diagrams we will use the formula (1.11).

Now we are ready to define the differential ∂ on the space of bracket star-diagrams.

Let A be a bracket star-diagram, and let t_α be one of its points without a star, then we define

$$\partial_{t_\alpha} A := P(A|_{x_{t_\alpha} = x_{t_{\alpha-}} \cdot x_{t_{\alpha+}}}), \quad (8.9)$$

where P is the projection of the space of generalized bracket star-diagrams on the space of bracket star-diagrams, that sends the space of special generalized bracket star-diagrams to zero; the points $t_{\alpha-}, t_{\alpha+} \in \mathbb{R}^1$ are respectively $t_\alpha - \epsilon$ and $t_\alpha + \epsilon$ for a very small $\epsilon > 0$.

Remark 8.10 The formula (8.9) can be made more precise:

$$\partial_{t_\alpha} A + (x_{t_{\alpha-}} - x_{t_{\alpha+}}) \cdot A = A|_{x_{t_\alpha} = x_{t_{\alpha-}} \cdot x_{t_{\alpha+}}}. \square \quad (8.11)$$

Let now t_β^* be one of the points of A having a star. We define

$$\partial_{t_\beta^*} A := P(A|_{x_{t_\beta^*} = x_{t_{\beta-}} \cdot x_{t_{\beta+}^*} + x_{t_{\beta-}^*} \cdot x_{t_{\beta+}} + [x_{t_{\beta-}}, x_{t_{\beta+}}]}), \quad (8.12)$$

where P is the same projection, the points $t_{\beta-}^{(*)}, t_{\beta+}^{(*)}$ are respectively $t_\beta^* - \epsilon$ and $t_\beta^* + \epsilon$ for a very small $\epsilon > 0$.

Remark 8.13 The formula (8.11) can be made more precise:

$$\partial_{t_\beta^*} A + (x_{t_{\beta-}} - x_{t_{\beta+}}) \cdot A = A|_{x_{t_\beta^*} = x_{t_{\beta-}} \cdot x_{t_{\beta+}^*} + x_{t_{\beta-}^*} \cdot x_{t_{\beta+}} + [x_{t_{\beta-}}, x_{t_{\beta+}}]}. \square \quad (8.14)$$

The differential ∂ on the space of bracket star-diagrams is the sum of the operators ∂_{t_α} and $\partial_{t_\beta^*}$ over all points $t_\alpha, \alpha \in \alpha$, and $t_\beta^*, \beta \in \beta$, of the corresponding (A, b) -configurations:

$$\partial = \sum_{\alpha \in \alpha} \partial_{t_\alpha} + \sum_{\beta \in \beta} \partial_{t_\beta^*}. \quad (8.15)$$

It is easy to see that $\partial^2 = 0$.

Remark 8.16

$$\partial A = \left(\sum_{\alpha \in \alpha} A|_{x_{t_\alpha}=x_{t_{\alpha-}} \wedge x_{t_{\alpha+}}} \right) + \left(\sum_{\beta \in \beta} A|_{x_{t_\beta^*}=x_{t_{\beta-}} \cdot x_{t_{\beta+}^*} + x_{t_{\beta-}} \cdot x_{t_{\beta+}} + [x_{t_{\beta-}}, x_{t_{\beta+}}]} \right) - (x_{t_-} - x_{t_+}) \cdot A, \quad (8.17)$$

where t_- (resp. t_+) is less (resp. greater) than all the points of the diagram A on the line \mathbb{R}^1 . \square

8.2 Case of even d

Let us fix an (A, b) -configuration J and consider the free Lie super-algebra with the even bracket and with the even generators x_{t_α} , $\alpha \in \alpha$, and the odd generators $x_{t_\beta^*}$, $\beta \in \beta$, where t_α , $\alpha \in \alpha$, and t_β^* , $\beta \in \beta$, are the points of our (A, b) -configuration J . Let us take the exterior algebra (in the super-sens) of the space of this Lie super-algebra. By convention the parity of an element $A = A_1 \wedge \dots \wedge A_k$ is $\tilde{A} = \tilde{A}_1 + \dots + \tilde{A}_k + k - 1$ the sum of the parities of A_i , $1 \leq i \leq k$, plus the number $k - 1$ of the exterior product signs. In the obtained space we will consider the subspace $\mathcal{BSD}(J)$ spanned by the analogous products of brackets (see the previous subsection 8.1). These products of brackets will be also called *bracket star-diagrams*. The *space of bracket star-diagrams* is defined analogously to the case of odd d . We also accept Definitions 8.4, 8.6, 8.7. Note that the parity of a (generalized) bracket star-diagram is the number of stars plus the number of the exterior product signs. This parity is opposite to the parity of the corresponding homology degree (8.2).

Let A and B be two generalized bracket star-diagrams, such that A can be inserted in the point $t_0^{(*)}$ of B . Let us define $B|_{x_{t_0^{(*)}}=A}$. To do this we replace $x_{t_0^{(*)}}$ in B by A , and we multiply the obtained expression by $(-1)^{(\tilde{A}-\epsilon_0) \times (n_1+n_2)}$, where ϵ_0 is equal to zero (resp. to one) if the point t_0 has no star (resp. if the point t_0^* has a star); n_1 (resp. n_2) is the number of the exterior product signs (resp. of the generators corresponding to stars) before $x_{t_0^{(*)}}$ in B .

Example 8.18

$$[x_{t_2} x_{t_3^*}] \wedge [x_{t_1^*} x_{t_0}] \Big|_{x_{t_0}=[x_{t_4} x_{t_5}] \wedge x_{t_6^*}} = (-1)^{(2-0) \cdot (1+2)} [x_{t_2} x_{t_3^*}] \wedge [x_{t_1^*}, [x_{t_4} x_{t_5}] \wedge x_{t_6^*}]. \quad \square$$

Let A be a bracket star-diagram, let t_α be a point of A without a star. Define

$$\partial_{t_\alpha} A := P \left(A|_{x_{t_\alpha}=x_{t_{\alpha-}} \wedge x_{t_{\alpha+}}} \right), \quad (8.19)$$

where P is the projection from the space of generalized bracket star-diagrams to the space of bracket star-diagrams.

Remark 8.20 The formula (8.19) can be made more precise:

$$\partial_{t_\alpha} A + (x_{t_{\alpha-}} - x_{t_{\alpha+}}) \wedge A = A|_{x_{t_\alpha}=x_{t_{\alpha-}} \wedge x_{t_{\alpha+}}}. \quad \square \quad (8.21)$$

Let t_β^* be a point of A having a star, we define

$$\partial_{t_\beta^*} A := P \left(A|_{x_{t_\beta^*}=x_{t_{\beta-}} \wedge x_{t_{\beta+}^*} - x_{t_{\beta-}} \wedge x_{t_{\beta+}} - [x_{t_{\beta-}}, x_{t_{\beta+}}]} \right). \quad \square \quad (8.22)$$

Remark 8.23 The formula (8.22) can be made more precise:

$$\partial_{t_\beta^*} A + (x_{t_{\beta-}} - x_{t_{\beta+}}) \wedge A = A|_{x_{t_\beta^*}=x_{t_{\beta-}} \wedge x_{t_{\beta+}^*}-x_{t_{\beta-}^*} \wedge x_{t_{\beta+}}-[x_{t_{\beta-}},x_{t_{\beta+}}]}. \quad \square \quad (8.24)$$

The differential ∂ on the space of bracket star-diagrams is defined by the formula (8.15) analogously to the case of odd d .

Remark 8.25

$$\partial A = \left(\sum_{\alpha \in \alpha} A|_{x_{t_\alpha}=x_{t_{\alpha-}} \wedge x_{t_{\alpha+}}} \right) + \left(\sum_{\beta \in \beta} A|_{x_{t_\beta^*}=x_{t_{\beta-}} \wedge x_{t_{\beta+}^*}-x_{t_{\beta-}^*} \wedge x_{t_{\beta+}}-[x_{t_{\beta-}},x_{t_{\beta+}}]} \right) - (x_{t_-} - x_{t_+}) \wedge A. \quad (8.26)$$

\square

9 Differential bialgebra of bracket star-diagrams

In this section we define the structure of differential bialgebras on the complexes of bracket star-diagrams. We conjecture that this structure is compatible with the corresponding bialgebra structure on the homology space of the long knots space; the corresponding conjectures are given in [T1, T2, T3].

The cases of odd and even d will be considered simultaneously.

Let D be a bracket star-diagram, T be a real number. Define a diagram D^T as the diagram obtained from D by the translation of \mathbb{R}^1 (D^T is equal to D):

$$t \mapsto t + T. \quad (9.1)$$

Let now A and B be two bracket star-diagrams, we define their product $A * B$ as the diagram $A^{-T} \cdot B^T$ in the case of odd d , and as the diagram $A^{-T} \wedge B^T$ in the case of even d , T being a very large positive number. This product resembles the product in the space of long knots:

$$\begin{array}{c} \text{---} \boxed{A} \text{---} * \text{---} \boxed{B} \text{---} = \text{---} \boxed{A} \text{---} \text{---} \boxed{B} \text{---} \end{array}$$

(Figure 9.2)

It follows from Theorem 3.6 and Corollary 5.7 that the homology algebra of the differential algebra of bracket star-diagrams is commutative in the case of even d for any commutative ring of coefficients. This is also true over \mathbb{Q} in the case of odd d , see Theorem 5.11.

Conjecture 9.3 Over \mathbb{Z} in the case of odd d , the homology algebra of the differential algebra of the bracket star-diagrams is not commutative. \square

Now we will define a comultiplication on the space of bracket star-diagrams. In the case of odd d the coproduct Δ of any diagram $A = A_1 \cdot A_2 \dots A_k$, where A_i , $1 \leq i \leq k$, are brackets, is defined as follows:

$$\Delta A = \Delta(A_1 \cdot A_2 \dots A_k) := \sum_{\substack{I \sqcup J = \{1, \dots, k\} \\ I = \{i_1 < \dots < i_l\} \\ J = \{j_1 < \dots < j_{k-l}\}}} (-1)^\epsilon A_{i_1} \cdot \dots \cdot A_{i_l} \otimes A_{j_1} \cdot \dots \cdot A_{j_{k-l}}, \quad (9.4)$$

Where $\epsilon = \sum_{i_p > j_q} \tilde{A}_{i_p} \tilde{A}_{j_q}$.

In the case of even d

$$\Delta(A) = \Delta(A_1 \wedge \dots \wedge A_k) := \sum_{\substack{I \cup J = \{1, \dots, k\} \\ I = \{i_1 < \dots < i_l\} \\ J = \{j_1 < \dots < j_{k-l}\}}} (-1)^\epsilon A_{i_1} \wedge \dots \wedge A_{i_l} \otimes A_{j_1} \wedge \dots \wedge A_{j_{k-l}}, \quad (9.5)$$

where $\epsilon = \sum_{i_p > j_q} (\tilde{A}_{i_p} + 1)(\tilde{A}_{j_q} + 1)$.

In other words, Δ is the standard symmetric coalgebra coproduct.

It can be easily verified that the operations Δ , $*$, ∂ define a differential bialgebra structure on the space of bracket star-diagrams.

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